

New Orlicz Affine Isoperimetric Inequalities *

Deping Ye

Abstract

The Orlicz-Brunn-Minkowski theory receives considerable attention recently, and many results in the L_p -Brunn-Minkowski theory have been extended to their Orlicz counterparts. The aim of this paper is to develop Orlicz L_ϕ affine and geominimal surface areas for single convex body as well as for multiple convex bodies, which generalize the L_p (mixed) affine and geominimal surface areas – fundamental concepts in the L_p -Brunn-Minkowski theory. Our extensions are different from the general affine surface areas by Ludwig (in Adv. Math. 224 (2010)). Moreover, our definitions for Orlicz L_ϕ affine and geominimal surface areas reveal that these affine invariants are essentially the infimum/supremum of $V_\phi(K, L^\circ)$, the Orlicz ϕ -mixed volume of K and the polar body of L , where L runs over all star bodies and all convex bodies, respectively, with volume of L equal to the volume of the unit Euclidean ball B_2^n . Properties for the Orlicz L_ϕ affine and geominimal surface areas, such as, affine invariance and monotonicity, are proved. Related Orlicz affine isoperimetric inequalities are also established.

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1 Introduction

The beautiful L_p -Brunn-Minkowski theory is developed by the combination of volume and the Firey p -sum of convex bodies (i.e., convex compact subsets in \mathbb{R}^n with nonempty interiors), and has found many applications such as in geometry. Key tools and objects include, for instance, L_p affine isoperimetric inequalities and L_p affine surface areas. L_p affine isoperimetric inequalities provide upper and/or lower bounds for (L_p) affine invariants defined on convex bodies in terms of volume, for example, the L_p affine isoperimetric inequalities for L_p centroid and projection bodies established in [29] (see also [9, 16, 28]). Lutwak, Yang and Zhang in [30, 31] extended such inequalities to their Orlicz counterparts, namely the affine isoperimetric inequalities for Orlicz centroid and projection bodies. These Orlicz affine isoperimetric inequalities initiated the study of the Orlicz-Brunn-Minkowski theory, which involves general (convex) functions and naturally generalizes the L_p -Brunn-Minkowski theory.

The literature for the Orlicz-Brunn-Minkowski theory expands quickly. Important contributions include the study of Logarithmic Minkowski and even Orlicz Minkowski problems [7, 15]; the log-Brunn-Minkowski-inequality [5]; stronger versions of the Orlicz-Petty projection inequality [6]; Orlicz Busemann-Petty centroid inequality [10, 20, 53] among others. Recently, in their seminal paper [13], Gardner, Hug and Weil built the foundation and provided a general framework for the Orlicz-Brunn-Minkowski theory. In particular, the Orlicz addition of convex bodies was proposed and the Orlicz mixed volume was defined. Moreover, they proved an Orlicz-Brunn-Minkowski

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inequality, which extends the classical Brunn-Minkowski inequality. As claimed in [13], “the Orlicz-Brunn-Minkowski theory is the most general possible based on an addition that retains all the basic geometrical properties enjoyed by the L_p -Brunn-Minkowski theory.”

In his ground breaking paper [27], Lutwak introduced the L_p affine surface areas for $p \geq 1$, which extend the classical affine surface area by Blaschke [4] in 1923. Note that, for $p \geq 1$ and for (smooth) convex body K , the L_p affine surface area of K , denoted by $as_p(K)$, has the following integral expression (see Section 2 for undefined notation)

$$as_p(K) = \int_{S^{n-1}} [h_K(u)^{1-p} f_K(u)]^{\frac{n}{n+p}} d\sigma(u). \quad (1.1)$$

This beautiful and convenient integral expression plays fundamental roles in extending the L_p affine surface area from $p \geq 1$ to all $-n \neq p \in \mathbb{R}$ (see e.g., [32, 39, 40]). The L_p affine surface area for $-n \neq p \in \mathbb{R}$ is affine invariant (i.e., invariant under all invertible linear transforms with unit absolute value of determinant), and is (essentially) the unique valuation with certain properties such as affine invariance and upper semi-continuity for $p > 0$ (see [22, 23] for more precise statements). Important applications of the L_p affine surface areas can be found in, e.g., [2, 3, 14, 18, 24, 34, 42, 43]. One of the most important results regarding the L_p affine surface area is its related L_p affine isoperimetric inequality (see e.g., [27, 44]): *Let convex body K have the origin as its centroid and have the same volume as the unit Euclidean ball B_2^n , then $as_p(K) \leq as_p(B_2^n)$ for $p > 0$ and $as_p(K) \geq as_p(B_2^n)$ for $-n < p < 0$, with equality if and only if K is an origin-symmetric ellipsoid.*

Lutwak also defined the L_p geominimal surface area for $p > 1$ in [27], a concept with similar properties to those for the L_p affine surface area. However, one *cannot* expect that the L_p geominimal surface area of K has similar integral expression to the formula (1.1) for its affine counterpart $as_p(K)$. In order to extend the L_p geominimal surface area from $p \geq 1$ to all $-n \neq p \in \mathbb{R}$, one turns to *observation (a): the L_p affine and geominimal surface areas of K for $p \geq 1$ are (essentially) the infimum of $V_p(K, L^\circ)$, the p -mixed volume of K and the polar body of L , where L runs over all star bodies and all convex bodies, respectively, with volume of L equal to the volume of the unit Euclidean ball B_2^n* (see [27] and formulas (2.7) and (2.9) for more details). In [47], the author proved similar formulas for the L_p affine surface areas for all $-n \neq p \in \mathbb{R}$, which motivate alternative definitions for the L_p affine surface areas of K and definitions for the L_p geominimal surface areas of K for all $-n \neq p \in \mathbb{R}$ (see formulas (2.7)-(2.10)). Contributions for L_p geominimal surface areas including related L_p affine isoperimetric inequalities can be found in, e.g., [27, 35, 36, 38, 47, 51].

Ludwig in [21] defined general affine surface areas involving general convex and concave functions, which are natural extensions of the L_p affine surface area based on formula (1.1). Affine invariance, the valuation property and affine isoperimetric inequalities for general affine surface areas are established in [21]. The monotone properties of general affine surface areas under the Steiner symmetrization were investigated in [48] and were used to prove stronger affine isoperimetric inequalities related to general affine surface areas. Again, due to lack of integral expressions for L_p geominimal surface areas, one *cannot* follow Ludwig’s ideas to define the Orlicz geominimal surface area.

This paper dedicates to develop the Orlicz L_ϕ affine and geominimal surface areas. Our motivations are the recent developed Orlicz mixed volume in [13] and *observation (a)*. As we can see in Definitions 3.1 and 3.2, our definitions for the Orlicz L_ϕ affine and geominimal surface areas are consistent with the *observation (a)*: *for instance, the Orlicz L_ϕ affine and geominimal surface areas for $\phi \in \Phi$ are (essentially) the infimum of $V_\phi(K, L^\circ)$, the Orlicz ϕ -mixed volume of K and the polar body of L , where L runs over all star bodies and all convex bodies, respectively, with*

volume of L equal to the volume of B_2^n . We prove the affine invariance and monotonicity for the Orlicz L_ϕ affine and geominimal surface areas. Moreover, we establish the following Orlicz affine isoperimetric inequalities.

Theorem 3.2 *Let K be a convex body with centroid at the origin and B_K be the origin-symmetric Euclidean ball with volume equal to the volume of K .*

(i) *For $\phi \in \Phi$, the following affine isoperimetric inequality holds, with equality if and only if K is an origin-symmetric ellipsoid,*

$$\Omega_\phi^{\text{orlicz}}(K) \leq G_\phi^{\text{orlicz}}(K) \leq G_\phi^{\text{orlicz}}([B_{K^\circ}]^\circ) = \Omega_\phi^{\text{orlicz}}([B_{K^\circ}]^\circ).$$

If in addition ϕ is increasing,

$$\Omega_\phi^{\text{orlicz}}(K) \leq G_\phi^{\text{orlicz}}(K) \leq G_\phi^{\text{orlicz}}(B_K) = \Omega_\phi^{\text{orlicz}}(B_K),$$

with equality if K is an origin-symmetric ellipsoid. Moreover, if ϕ is strictly increasing, equality holds if and only if K is an origin-symmetric ellipsoid.

(ii) *For $\phi \in \Psi$, the following affine isoperimetric inequality holds with equality if K is an origin-symmetric ellipsoid,*

$$\Omega_\phi^{\text{orlicz}}(K) \geq G_\phi^{\text{orlicz}}(K) \geq G_\phi^{\text{orlicz}}(B_K) = \Omega_\phi^{\text{orlicz}}(B_K).$$

If in addition ϕ is strictly decreasing, equality holds if and only if K is an origin-symmetric ellipsoid.

In Section 4, we provide definitions for the Orlicz mixed L_ϕ affine and geominimal surface areas as well as the Orlicz i -th mixed L_ϕ affine and geominimal surface areas. We briefly discuss their properties including affine invariance, Alexander-Fenchel type inequality, and affine isoperimetric inequalities. Basic background and notation in convex geometry are provided in Section 2 and readers can read the nice book [37] by Schneider for more background in convex geometry.

2 Background and Notation

Throughout this paper, \mathcal{K} denotes the set of n -dimensional *convex bodies* in \mathbb{R}^n : convex compact subsets of \mathbb{R}^n with nonempty interior. Write \mathcal{K}_0 and \mathcal{K}_c for the subsets of convex bodies with the origin in their interiors and with centroid at the origin respectively. Denote by B_2^n the unit Euclidean ball and by S^{n-1} the unit sphere in \mathbb{R}^n . The volume of B_2^n is written by ω_n . We use $SL(n)$ to denote the group of special linear transformations from \mathbb{R}^n to \mathbb{R}^n . That is, $T \in SL(n)$ means that T is a linear transform with $|\det(T)| = 1$, where $|\det(T)|$ refers to the absolute value of the determinant of T . The transpose and the inverse of T will be denoted by T^* and T^{-1} respectively. We often write TK for $T(K)$ with $K \in \mathcal{K}_0$. A convex body $\mathcal{E} \in \mathcal{K}_0$ is said to be an *origin-symmetric ellipsoid* if $\mathcal{E} = rTB_2^n$ for some $r > 0$ and $T \in SL(n)$.

Both support function and radial function can be used to uniquely determine a convex body. The *support function* of $K \in \mathcal{K}_0$, $h_K : S^{n-1} \rightarrow (0, \infty)$, is defined as $h_K(u) = \max_{x \in K} \langle x, u \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathbb{R}^n and induces the usual Euclidian norm $\|\cdot\|$. The *radial function* of $K \in \mathcal{K}_0$, $\rho_K : S^{n-1} \rightarrow (0, \infty)$, is formulated by

$$\rho_K(u) = \max\{\lambda : \lambda u \in K\}. \quad (2.2)$$

In fact, formula (2.2) can be used to define the radial function for star bodies. We use \mathcal{S}_0 to denote the set of star bodies (about the origin) in \mathbb{R}^n . That is, $L \in \mathcal{S}_0$ means that the line segment from

0 to any point $x \in L$ is contained in L , and the radial function of L , $\rho_L(\cdot)$ defined by formula (2.2), is continuous and positive. Clearly, $\mathcal{K}_0 \subset \mathcal{S}_0$.

For $K \in \mathcal{K}_0$, one can define K° , the *polar body* of K , by $K^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall x \in K\}$. Note that $K^\circ \in \mathcal{K}_0$ and $(K^\circ)^\circ = K$ for $K \in \mathcal{K}_0$. Moreover, if $K \in \mathcal{K}_0$, then $\rho_K(u)h_{K^\circ}(u) = 1$ holds for all $u \in S^{n-1}$. Write $|K|$ for the volume of K and for the Hausdorff content of its appropriate dimension if K is a general subset of \mathbb{R}^n . Clearly for all $L \in \mathcal{S}_0$ and all $K \in \mathcal{K}_0$, one has

$$|L| = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^n d\sigma(u) \quad \text{and} \quad |K^\circ| = \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_K^n(u)} d\sigma(u),$$

where σ is the usual spherical measure on S^{n-1} . Denote by $\mathcal{K}_s \subset \mathcal{K}_0$ the set of convex bodies with Santaló point at the origin, i.e., $K \in \mathcal{K}_s \Leftrightarrow K^\circ \in \mathcal{K}_c$.

The volume radius of K , denoted by $\text{vrad}(K)$, is a way to measure the size of $K \in \mathcal{K}_0$ in terms of volume. It takes the following form

$$\text{vrad}(K) = \left(\frac{|K|}{|B_2^n|} \right)^{1/n} \iff |K|^{1/n} = \omega_n^{1/n} \text{vrad}(K). \quad (2.3)$$

Clearly, $\text{vrad}(rB_2^n) = r$ for all $r > 0$, and for all $T \in SL(n)$,

$$\text{vrad}(TK) = \text{vrad}(K). \quad (2.4)$$

It is well known that there is a universal (independent of K and n) constant $c > 0$, such that, $c \leq \text{vrad}(K)\text{vrad}(K^\circ) \leq 1$ for all $K \in \mathcal{K}_c$ (or $K \in \mathcal{K}_s$). The upper bound (i.e., the celebrated Blaschke-Santaló inequality) is tight, and equality holds if and only if K is an origin-symmetric ellipsoid. The lower bound is the famous Bourgain-Milman inverse Santaló inequality [8]. Estimates on the constant c can be found in [19, 33]. Note that finding the precise minimal value for $\text{vrad}(K)\text{vrad}(K^\circ)$ is still an open problem and is known as the Mahler conjecture: among all origin-symmetric convex bodies (i.e., $K = -K$), the cube is conjectured to be a minimizer for $\text{vrad}(K)\text{vrad}(K^\circ)$; while the simplex is conjectured to be a minimizer for $\text{vrad}(K)\text{vrad}(K^\circ)$ among all convex bodies $K \in \mathcal{K}_c$ (or $K \in \mathcal{K}_s$).

The *Firey p -sum* [12] of $K, L \in \mathcal{K}_0$ with $\lambda, \eta \geq 0$ (not both zeros) for $p \geq 1$, denoted by $\lambda K +_p \eta L$, is determined by the support function

$$(h_{\lambda K +_p \eta L}(u))^p = \lambda (h_K(u))^p + \eta (h_L(u))^p.$$

Lutwak [26] defined $V_p(K, L)$, the *p -mixed volume* of K and L , by

$$\frac{1}{p} V_p(K, L) = \lim_{\epsilon \rightarrow 0} \frac{|K +_p \epsilon L| - |K|}{n\epsilon},$$

and proved that there is a measure $S_p(K, \cdot)$, such that

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_p(K, u).$$

Note that $V_1(K, L)$ for $p = 1$ is the classical *mixed volume* of K and L . It has been proved that there is a positive Borel measure $S(K, \cdot)$ on S^{n-1} (see [1, 11]), such that, for $K, L \in \mathcal{K}$,

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u) dS(K, u).$$

The relation between $S(K, \cdot)$ and $S_p(K, \cdot)$ for all $p \geq 1$ is

$$dS_p(K, u) = h_K^{1-p}(u) dS(K, u).$$

One can actually define the p -mixed volume (see [47]) for all $p \in \mathbb{R}$ by

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u)^p h_K^{1-p}(u) dS(K, u). \quad (2.5)$$

When $K \in \mathcal{K}_0$ and $L \in \mathcal{S}_0$, we use the following formula

$$V_p(K, L^\circ) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^{-p} h_K^{1-p}(u) dS(K, u), \quad p \in \mathbb{R}.$$

It is easy to see that for all $\lambda > 0$,

$$V_p(K, \lambda L) = \lambda^p V_p(K, L). \quad (2.6)$$

We say that $K \in \mathcal{K}$ has *curvature function* $f_K(\cdot)$ if the measure $S(K, \cdot)$ is absolutely continuous with respect to the spherical measure σ and satisfies $dS(K, u) = f_K(u) d\sigma(u)$. Let $\mathcal{F}_0 \subset \mathcal{K}_0$ be the subset of convex bodies with curvature function and with the origin in its interior. Also, let $\mathcal{F}_c = \mathcal{F}_0 \cap \mathcal{K}_c$ and $\mathcal{F}_s = \mathcal{F}_0 \cap \mathcal{K}_s$. The set \mathcal{F}_0^+ denotes the subset of convex bodies in \mathcal{F}_0 with continuous positive curvature functions.

The L_p affine surface area of K , $as_p(K)$, is a fundamental object in affine convex geometry. It can be formulated by (see, e.g., [27, 47])

$$as_p(K) = \inf_{L \in \mathcal{S}_0} \left\{ n V_p(K, L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\}, \quad p \geq 0; \quad (2.7)$$

$$as_p(K) = \sup_{L \in \mathcal{S}_0} \left\{ n V_p(K, L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\}, \quad -n \neq p < 0. \quad (2.8)$$

The L_p affine surface area has many nice properties such as the valuation property. Moreover, for all invertible linear transforms $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, one has

$$as_p(TK) = |\det(T)|^{\frac{n-p}{n+p}} as_p(K).$$

The L_p geominimal surface area of K , $\tilde{G}_p(K)$, can be defined by [47]

$$\tilde{G}_p(K) = \inf_{Q \in \mathcal{K}_0} \left\{ n V_p(K, Q^\circ)^{\frac{n}{n+p}} |Q|^{\frac{p}{n+p}} \right\}, \quad p \geq 0; \quad (2.9)$$

$$\tilde{G}_p(K) = \sup_{Q \in \mathcal{K}_0} \left\{ n V_p(K, Q^\circ)^{\frac{n}{n+p}} |Q|^{\frac{p}{n+p}} \right\}, \quad -n \neq p < 0. \quad (2.10)$$

Note that for all invertible linear transforms $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, one also has

$$\tilde{G}_p(TK) = |\det(T)|^{\frac{n-p}{n+p}} \tilde{G}_p(K).$$

Recall that Lutwak defined the L_p geominimal surface area of K for $p \geq 1$ in [27] by

$$\omega_n^{p/n} G_p(K) = \inf_{Q \in \mathcal{K}_0} \{ n V_p(K, Q) |Q^\circ|^{p/n} \}.$$

Thus, the relation between $\tilde{G}_p(K)$ and $G_p(K)$ for $p \geq 1$ is

$$[\tilde{G}_p(K)]^{n+p} = (n\omega_n)^p [G_p(K)]^n. \quad (2.11)$$

In fact, one has the following formulas for the L_p geominimal surface areas:

$$(\tilde{G}_p(K))^{\frac{n+p}{n}} = \inf_{Q \in \mathcal{K}_0} \left\{ n^{\frac{n+p}{n}} V_p(K, Q^\circ) |Q|^{\frac{p}{n}} \right\}, \quad p \geq 0 \text{ \& } p < -n; \quad (2.12)$$

$$(\tilde{G}_p(K))^{\frac{n+p}{n}} = \sup_{Q \in \mathcal{K}_0} \left\{ n^{\frac{n+p}{n}} V_p(K, Q^\circ) |Q|^{\frac{p}{n}} \right\}, \quad -n < p < 0. \quad (2.13)$$

Similar formulas for L_p affine surface areas can also be obtained. Based on formulas (2.12) and (2.13), we have the following *observation (b): the convexity and concavity of function $t^{-p/n}$ (not t^p itself) determines the supremum and the infimum*. More precisely, formula (2.12) is related to $t^{-p/n}$ being convex for $p \geq 0$ and $p < -n$; while formula (2.13) is related to $t^{-p/n}$ being concave for $-n < p < 0$.

We will work on function $\phi(t) : (0, \infty) \rightarrow (0, \infty)$. Such a function is said to be convex if

$$\phi(\lambda t + (1 - \lambda)s) \leq \lambda \phi(t) + (1 - \lambda)\phi(s), \quad \forall t, s \in (0, \infty) \text{ and } \lambda \in [0, 1].$$

Function $\phi(t)$ is said to be strictly convex if

$$\phi(\lambda t + (1 - \lambda)s) < \lambda \phi(t) + (1 - \lambda)\phi(s), \quad \forall t, s \in (0, \infty) \text{ with } t \neq s, \text{ and } \lambda \in (0, 1).$$

Similarly, one can define concave and strictly concave function $\phi(t)$ by changing the directions of above inequalities. Function ϕ is increasing if $\phi(t) \leq \phi(s)$ and is strictly increasing if $\phi(t) < \phi(s)$ for all $t < s$; while function ϕ is decreasing if $\phi(t) \geq \phi(s)$ and is strictly decreasing if $\phi(t) > \phi(s)$ for all $t < s$. The inverse function of ϕ , if exists, is denoted by $\phi^{-1}(t)$.

3 Orlicz L_ϕ affine and geominimal surface areas

Let $\phi : (0, \infty) \rightarrow (0, \infty)$ be a positive continuous function. Define the Orlicz ϕ -mixed volume $V_\phi(K, Q)$ of convex bodies $K, Q \in \mathcal{K}_0$ by

$$V_\phi(K, Q) = \frac{1}{n} \int_{S^{n-1}} \phi \left(\frac{h_Q(u)}{h_K(u)} \right) h_K(u) dS(K, u).$$

When $K \in \mathcal{K}_0$ and $L \in \mathcal{S}_0$, we use the following formula

$$V_\phi(K, L^\circ) = \frac{1}{n} \int_{S^{n-1}} \phi \left(\frac{1}{\rho_L(u) h_K(u)} \right) h_K(u) dS(K, u).$$

For $\phi(t) = t^p$ with $p \in \mathbb{R}$, one gets the p -mixed volume $V_p(K, Q)$ given by formula (2.5). If ϕ is a convex function satisfies $\phi(0) = 0$ and $\phi(1) = 1$, our Orlicz ϕ -mixed volume $V_\phi(K, Q)$ is identical to the one introduced in [13].

For function $\phi : (0, \infty) \rightarrow (0, \infty)$, define $F_\phi(t) = \phi(t^{-1/n})$. For simplicity, we often write $F(t)$ for $F_\phi(t)$ and clearly $\phi(t) = F(t^{-n})$. From observation (b), one sees that it is the convexity and concavity of function $F(t)$ (not $\phi(t)$ itself) determining the supremum and the infimum. This observation is further strengthened by the proof of Corollary 3.1. Clearly, if $\phi(t)$ is increasing then $F(t)$ is decreasing; while if $\phi(t)$ is decreasing then $F(t)$ is increasing. Vice versa, if $F(t)$ is increasing then $\phi(t)$ is decreasing; while if $F(t)$ is decreasing then $\phi(t)$ is increasing. The relation of convexity and concavity between $\phi(t)$ and $F(t)$ is not clear. However, if $\phi(t)$ is convex and increasing, then $F(t)$ is convex and decreasing. To see this, let $t, s \in (0, \infty)$ and $\lambda \in [0, 1]$, then

$$\begin{aligned} F(\lambda t + (1 - \lambda)s) &= \phi([\lambda t + (1 - \lambda)s]^{-1/n}) \leq \phi(\lambda t^{-1/n} + (1 - \lambda)s^{-1/n}) \\ &\leq \lambda \phi(t^{-1/n}) + (1 - \lambda)\phi(s^{-1/n}) = \lambda F(t) + (1 - \lambda)F(s), \end{aligned}$$

where the first inequality follows from the convexity of $t^{-1/n}$ and the increasing property of $\phi(t)$, and the second inequality follows from the convexity of $\phi(t)$. Similarly, if $F(t)$ is convex and increasing, then $\phi(t)$ is convex and decreasing.

Let the set of functions Φ be

$$\Phi = \{\phi : (0, \infty) \rightarrow (0, \infty) : F(t) \text{ is either a constant or a strictly convex function}\}.$$

The set Φ contains functions such as t^p with $p \in (-\infty, -n) \cup (0, \infty)$ and all convex increasing functions. Note that Φ could have neither convex nor concave functions, such as, $e^{-(t^{-n})}$ whose second order derivative is $e^{-(t^{-n})}nt^{-n-2}(nt^{-n} - n - 1)$ and changes the sign at (the inflection point) $t = \left(\frac{n}{n+1}\right)^{1/n}$. One may define even more general set than the above set Φ by changing “strictly convex” to convex; however, the strict convexity of $F(t)$ required in the definition of Φ is mainly to exclude those functions ϕ proportional to t^{-n} or even part of $\phi(t)$ proportional to t^{-n} , a function with problems in defining related affine and geominimal surface areas. (Note that L_p affine and geominimal surface areas were defined for functions $\phi(t) = t^p$ with $p \neq -n$, see formulas (2.7), (2.8), (2.9) and (2.10)).

Note that if a concave function $F : (0, \infty) \rightarrow (0, \infty)$ is decreasing on $[t_0, \infty)$ for some $t_0 > 0$, then $F(t) = F(t_0)$ for all $t > t_0$. To this end, one first gets that $\lim_{t \rightarrow \infty} F(t)$ exists and is finite, say equal to $a \geq 0$. On the other hand, let $\lambda = \frac{t-2t_0}{2t-2t_0}$ for $t > 2t_0$, one has $F(t/2) = F(\lambda t + (1-\lambda)t_0) \geq \lambda F(t) + (1-\lambda)F(t_0)$. Taking limit with $t \rightarrow \infty$, one gets $\lambda \rightarrow 1/2$ and hence $a \geq a/2 + F(t_0)/2 \Leftrightarrow a \geq F(t_0)$. This further implies that $a = F(t_0)$ as $F(t)$ is decreasing, which leads to $F(t) = a$ for all $t \geq t_0$. In view of this, let the set of functions Ψ be

$$\Psi = \{\phi : (0, \infty) \rightarrow (0, \infty) : F(t) \text{ is either a constant or an increasing strictly concave function}\}.$$

Clearly, ϕ is decreasing and so is (if exists) $\phi^{-1}(t)$ for $\phi \in \Psi$. Sample (non-constant) functions in Ψ are: t^p with $p \in (-n, 0)$, $\arctan(t^{-n})$, and $\ln(1 + t^{-n})$. Note that Ψ could have neither convex nor concave functions, such as, $\arctan(t^{-n})$ whose second order derivative is $nt^{-n-2}(1 + t^{-2n})^{-2}[(n+1) - (n-1)t^{-2n}]$ and changes the sign at (the inflection point) $t = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2n}}$. One may define even more general set than the above set Ψ by changing “strictly concave” to concave; however, the strict concavity of $F(t)$ required in the definition of Ψ is mainly to exclude those functions ϕ proportional to t^{-n} or even part of $\phi(t)$ proportional to t^{-n} . We also mention that both Φ and Ψ may not contain some nice functions such as $\phi(t) = e^{-t}$.

We now propose the following definition for the Orlicz L_ϕ affine surface area.

Definition 3.1 Let $K \in \mathcal{K}_0$ be a convex body with the origin in its interior.

(i) For $\phi \in \Phi$, we define the Orlicz L_ϕ affine surface area of K by

$$\Omega_\phi^{orlicz}(K) = \inf_{L \in \mathcal{S}_0} \{nV_\phi(K, \text{vrad}(L)L^\circ)\} = \inf \{nV_\phi(K, L^\circ) : L \in \mathcal{S}_0 \text{ with } |L| = \omega_n\}.$$

(ii) For $\phi \in \Psi$, we define the Orlicz L_ϕ affine surface area of K by

$$\Omega_\phi^{orlicz}(K) = \sup_{L \in \mathcal{S}_0} \{nV_\phi(K, \text{vrad}(L)L^\circ)\} = \sup \{nV_\phi(K, L^\circ) : L \in \mathcal{S}_0 \text{ with } |L| = \omega_n\}.$$

Remark. Clearly, if $\phi(t) = a > 0$ is a constant function, then $\Omega_\phi^{orlicz}(K) = an|K|$. We write $\Omega_p^{orlicz}(K)$ for the case $\phi(t) = t^p$ with $-n \neq p \in \mathbb{R}$ and in fact

$$(n\omega_n)^{p/n}\Omega_p^{orlicz}(K) = (as_p(K))^{\frac{n+p}{n}},$$

where $as_p(K)$ is the L_p affine surface area of K defined in (2.7) and (2.8). As an example, we show this by letting $\phi(t) = t^p$ with $p \geq 0$. Then

$$\begin{aligned} (n\omega_n)^{p/n} \Omega_p^{orlicz}(K) &= (n\omega_n)^{p/n} \inf_{L \in \mathcal{S}_0} \{nV_p(K, \text{vrad}(L)L^\circ)\} \\ &= n^{\frac{n+p}{n}} \inf_{L \in \mathcal{S}_0} \left\{ V_p(K, L^\circ) [\omega_n^{1/n} \text{vrad}(L)]^p \right\} \\ &= \left(\inf_{L \in \mathcal{S}_0} \left\{ nV_p(K, L^\circ)^{\frac{n}{n+p}} |L|^{\frac{p}{n+p}} \right\} \right)^{\frac{n+p}{n}} = (as_p(K))^{\frac{n+p}{n}}, \end{aligned}$$

where we have used formula (2.6) in the second equality and formula (2.3) in the third equality.

We propose the following definition for the Orlicz L_ϕ geominimal surface area.

Definition 3.2 Let $K \in \mathcal{K}_0$ be a convex body with the origin in its interior.

(i) For $\phi \in \Phi$, we define the Orlicz L_ϕ geominimal surface area of K by

$$G_\phi^{orlicz}(K) = \inf_{Q \in \mathcal{K}_0} \{nV_\phi(K, \text{vrad}(Q^\circ)Q)\} = \inf \{nV_\phi(K, Q) : Q \in \mathcal{K}_0 \text{ with } |Q^\circ| = \omega_n\}. \quad (3.14)$$

(ii) For $\phi \in \Psi$, we define the Orlicz L_ϕ geominimal surface area of K by

$$G_\phi^{orlicz}(K) = \sup_{Q \in \mathcal{K}_0} \{nV_\phi(K, \text{vrad}(Q^\circ)Q)\} = \sup \{nV_\phi(K, Q) : Q \in \mathcal{K}_0 \text{ with } |Q^\circ| = \omega_n\}. \quad (3.15)$$

Remark. Clearly, if $\phi(t) = a > 0$ is a constant function, then $G_\phi^{orlicz}(K) = an|K|$. We write $G_p^{orlicz}(K)$ for the case $\phi(t) = t^p$ with $-n \neq p \in \mathbb{R}$ and in fact, similar to the remark after Definition 3.1,

$$(n\omega_n)^{p/n} G_p^{orlicz}(K) = (\tilde{G}_p(K))^{\frac{n+p}{n}},$$

where $\tilde{G}_p(K)$ is given by formulas (2.9) and (2.10). Formula (2.11) further implies $G_p^{orlicz}(K) = G_p(K)$ for $p \geq 1$, i.e., our Orlicz L_ϕ geominimal surface area for $\phi(t) = t^p$ with $p \geq 1$ is identical to the one by Lutwak.

Proposition 3.1 Let $K \in \mathcal{K}_0$. Assume that $\phi \leq \psi$ with either $\phi, \psi \in \Phi$ or $\phi, \psi \in \Psi$, then

$$\Omega_\phi^{orlicz}(K) \leq \Omega_\psi^{orlicz}(K) \quad \text{and} \quad G_\phi^{orlicz}(K) \leq G_\psi^{orlicz}(K).$$

Proof. We only prove the case for the Orlicz L_ϕ geominimal surface area with $\phi, \psi \in \Phi$ such that $\phi \leq \psi$. Other cases can be proved along the same line.

In fact, for all $K, Q \in \mathcal{K}_0$, one has

$$\phi\left(\frac{h_Q(u)}{h_K(u)}\right) \leq \psi\left(\frac{h_Q(u)}{h_K(u)}\right), \quad \forall u \in S^{n-1}.$$

Hence, $V_\phi(K, Q) \leq V_\psi(K, Q)$ for all $K, Q \in \mathcal{K}_0$. Formula (3.14) implies that $G_\phi^{orlicz}(K) \leq G_\psi^{orlicz}(K)$ for $\phi \leq \psi$ with $\phi, \psi \in \Phi$.

The following proposition states that both Orlicz L_ϕ affine and geominimal surface areas are affine invariant.

Proposition 3.2 Let $K \in \mathcal{K}_0$. For all $\phi \in \Phi$ or $\phi \in \Psi$, one has

$$\Omega_\phi^{orlicz}(TK) = \Omega_\phi^{orlicz}(K); \quad G_\phi^{orlicz}(TK) = G_\phi^{orlicz}(K), \quad \forall T \in SL(n).$$

Proof. Let $T \in SL(n)$ and $u = u(v) = \frac{T^*v}{\|T^*v\|} \in S^{n-1}$ for $v \in S^{n-1}$. Note that $\frac{1}{n}h_K(u) dS(K, u)$ is the volume element of K and hence $h_{TK}(v) dS(TK, v) = h_K(u) dS(K, u)$. On the other hand, $\forall y \in TK$, there is a unique $x \in K$ s.t. $y = Tx$ (as T is invertible). Thus,

$$h_{TK}(v) = \max_{y \in TK} \langle y, v \rangle = \max_{x \in K} \langle Tx, v \rangle = \max_{x \in K} \langle x, T^*v \rangle = \|T^*v\| \max_{x \in K} \langle x, u \rangle = \|T^*v\| h_K(u), \quad (3.16)$$

which implies that, for $T \in SL(n)$,

$$nV_\phi(TK, TQ) = \int_{S^{n-1}} \phi\left(\frac{h_{TQ}(v)}{h_{TK}(v)}\right) h_{TK}(v) dS(TK, v) = \int_{S^{n-1}} \phi\left(\frac{h_Q(u)}{h_K(u)}\right) h_K(u) dS(K, u) = nV_\phi(K, Q).$$

Together with formula (2.4), one gets, for all $\phi \in \Phi$,

$$G_\phi^{orlicz}(TK) = \inf_{Q \in \mathcal{K}_0} \{nV_\phi(TK, \text{vrad}((TQ)^\circ)(TQ))\} = \inf_{Q \in \mathcal{K}_0} \{nV_\phi(K, \text{vrad}(Q^\circ)Q)\} = G_\phi^{orlicz}(K);$$

while for all $\phi \in \Psi$,

$$G_\phi^{orlicz}(TK) = \sup_{Q \in \mathcal{K}_0} \{nV_\phi(TK, \text{vrad}((TQ)^\circ)(TQ))\} = \sup_{Q \in \mathcal{K}_0} \{nV_\phi(K, \text{vrad}(Q^\circ)Q)\} = G_\phi^{orlicz}(K).$$

The proof for the case $\Omega_\phi^{orlicz}(TK) = \Omega_\phi^{orlicz}(K)$ follows along the same line if one notices that, by similar calculation to formula (3.16), $\rho_{(T^*)^{-1}L}(v)\|T^*v\| = \rho_L(u)$ and therefore

$$\begin{aligned} nV_\phi(TK, ((T^*)^{-1}L)^\circ) &= \int_{S^{n-1}} \phi\left(\frac{1}{\rho_{(T^*)^{-1}L}(v)h_{TK}(v)}\right) h_{TK}(v) dS(TK, v) \\ &= \int_{S^{n-1}} \phi\left(\frac{1}{\rho_L(u)h_K(u)}\right) h_K(u) dS(K, u) = nV_\phi(K, L^\circ). \end{aligned}$$

Remark. When $\phi(t) = t^p$ with $-n \neq p \in \mathbb{R}$, a more careful calculation shows that

$$G_p^{orlicz}(TK) = |\det(T)|^{\frac{n-p}{n}} G_p^{orlicz}(K); \quad \Omega_p^{orlicz}(TK) = |\det(T)|^{\frac{n-p}{n}} \Omega_p^{orlicz}(K).$$

Moreover, if $p = 1$, both $\Omega_p^{orlicz}(K)$ and $G_p^{orlicz}(K)$ are translation invariant, i.e., $\forall z_0 \in \mathbb{R}^n$,

$$G_p^{orlicz}(K - z_0) = G_p^{orlicz}(K); \quad \Omega_p^{orlicz}(K - z_0) = \Omega_p^{orlicz}(K), \text{ for } p = 1.$$

However, for general $\phi \in \Phi$ or $\phi \in \Psi$, one *cannot* expect the translation invariance even for K being ellipsoids, see inequalities (3.23) and (3.25) for special cases.

Proposition 3.3 *Let $K \in \mathcal{K}_0$ be a convex body in \mathbb{R}^n with the origin in its interior.*

(i) *For $\phi \in \Phi$, one has*

$$\Omega_\phi^{orlicz}(K) \leq G_\phi^{orlicz}(K).$$

(ii) *For $\phi \in \Psi$, one has*

$$\Omega_\phi^{orlicz}(K) \geq G_\phi^{orlicz}(K).$$

Proof. (i). Note that $\mathcal{K}_0 \subset \mathcal{S}_0$. Therefore, for $\phi \in \Phi$, one has

$$\Omega_\phi^{orlicz}(K) = \inf_{L \in \mathcal{S}_0} \{nV_\phi(K, \text{vrad}(L)L^\circ)\} \leq \inf_{Q \in \mathcal{K}_0} \{nV_\phi(K, \text{vrad}(Q)Q^\circ)\} = G_\phi^{orlicz}(K).$$

(ii). For $\phi \in \Psi$, one gets

$$\Omega_\phi^{orlicz}(K) = \sup_{L \in \mathcal{S}_0} \{nV_\phi(K, \text{vrad}(L)L^\circ)\} \geq \sup_{Q \in \mathcal{K}_0} \{nV_\phi(K, \text{vrad}(Q)Q^\circ)\} = G_\phi^{orlicz}(K).$$

Corollary 3.1 *Let \mathcal{E} be an origin-symmetric ellipsoid. For $\phi \in \Phi$ or $\phi \in \Psi$, one has*

$$\Omega_\phi^{orlicz}(\mathcal{E}) = G_\phi^{orlicz}(\mathcal{E}) = n\phi(\text{vrad}(\mathcal{E}^\circ))|\mathcal{E}|.$$

Proof. Let us first calculate $\Omega_\phi^{orlicz}(rB_2^n)$ for some $r > 0$. Taking $Q = B_2^n$ (and hence $\text{vrad}(B_2^n) = 1$) in formula (3.14), one has, for $\phi \in \Phi$,

$$\Omega_\phi^{orlicz}(rB_2^n) \leq G_\phi^{orlicz}(rB_2^n) \leq nV_\phi(rB_2^n, B_2^n) = \int_{S^{n-1}} \phi(1/r) r^n d\sigma(u) = n\phi(1/r)|rB_2^n|.$$

On the other hand, Proposition 3.3 implies

$$\begin{aligned} G_\phi^{orlicz}(rB_2^n) &\geq \Omega_\phi^{orlicz}(rB_2^n) = n \inf_{L \in \mathcal{S}_0} V_\phi(rB_2^n, \text{vrad}(L)L^\circ) \\ &= nr^n \inf_{L \in \mathcal{S}_0} \left[\frac{1}{n} \int_{S^{n-1}} F\left(\frac{r^n \rho_L^n(u)|B_2^n|}{|L|}\right) d\sigma(u) \right] \\ &\geq n|rB_2^n| \inf_{L \in \mathcal{S}_0} F\left(r^n \cdot \frac{1}{n|L|} \int_{S^{n-1}} \rho_L^n(u) d\sigma(u)\right) \\ &= nF(r^n)|rB_2^n| = n\phi(1/r)|rB_2^n|, \end{aligned}$$

where the second inequality follows from Jensen's inequality (for convex function F as $\phi \in \Phi$). In conclusion, for $\phi \in \Phi$,

$$G_\phi^{orlicz}(rB_2^n) = \Omega_\phi^{orlicz}(rB_2^n) = n\phi(1/r)|rB_2^n| = n\phi(\text{vrad}((rB_2^n)^\circ)) \cdot |rB_2^n|.$$

Similarly, for $\phi \in \Psi$, one has

$$\begin{aligned} n\phi(1/r)|rB_2^n| &\leq G_\phi^{orlicz}(rB_2^n) \leq \Omega_\phi^{orlicz}(rB_2^n) = n \sup_{L \in \mathcal{S}_0} V_\phi(rB_2^n, \text{vrad}(L)L^\circ) \\ &\leq n|rB_2^n| \sup_{L \in \mathcal{S}_0} F\left(\frac{1}{n} \int_{S^{n-1}} \frac{r^n \rho_L^n(u)}{|L|} d\sigma(u)\right) = n\phi(1/r)|rB_2^n|, \end{aligned}$$

where the last inequality follows from Jensen's inequality (for concave function F as $\phi \in \Psi$). In conclusion, for $\phi \in \Psi$, one also has

$$G_\phi^{orlicz}(rB_2^n) = \Omega_\phi^{orlicz}(rB_2^n) = n\phi(1/r)|rB_2^n| = n\phi(\text{vrad}((rB_2^n)^\circ)) \cdot |rB_2^n|.$$

Let \mathcal{E} be any origin-symmetric ellipsoid with $|\mathcal{E}| = |rB_2^n|$ for some $r > 0$. Then, $\mathcal{E} = T(rB_2^n)$ for some $T \in SL(n)$. Proposition 3.2 implies that, for $\phi \in \Phi$ or $\phi \in \Psi$,

$$\begin{aligned} \Omega_\phi^{orlicz}(\mathcal{E}) &= \Omega_\phi^{orlicz}(rB_2^n) = n\phi(\text{vrad}((rB_2^n)^\circ)) \cdot |rB_2^n| = n|\mathcal{E}| \cdot \phi(\text{vrad}(\mathcal{E}^\circ)); \\ G_\phi^{orlicz}(\mathcal{E}) &= G_\phi^{orlicz}(rB_2^n) = n\phi(\text{vrad}((rB_2^n)^\circ)) \cdot |rB_2^n| = n|\mathcal{E}| \cdot \phi(\text{vrad}(\mathcal{E}^\circ)). \end{aligned}$$

In particular, for all \mathcal{E} with $|\mathcal{E}| = |B_2^n|$, we have

$$G_\phi^{orlicz}(\mathcal{E}) = \Omega_\phi^{orlicz}(\mathcal{E}) = n|\mathcal{E}| \cdot \phi(1).$$

The following proposition compares the Orlicz L_ϕ affine and geominimal surface areas with the Orlicz ϕ -surface area of K defined by $S_\phi(K) = nV_\phi(K, B_2^n)$. Clearly,

$$S_\phi(B_2^n) = nV_\phi(B_2^n, B_2^n) = \int_{S^{n-1}} \phi(1) d\sigma(u) = \phi(1) \cdot n|B_2^n| = G_\phi^{orlicz}(B_2^n) = \Omega_\phi^{orlicz}(B_2^n).$$

Proposition 3.4 Let $K \in \mathcal{K}_0$ be a convex body in \mathbb{R}^n with the origin in its interior.

(i) For $\phi \in \Phi$, one has

$$\Omega_\phi^{orlicz}(K) \leq G_\phi^{orlicz}(K) \leq S_\phi(K).$$

(ii) For $\phi \in \Psi$, one has

$$\Omega_\phi^{orlicz}(K) \geq G_\phi^{orlicz}(K) \geq S_\phi(K).$$

Proof. (i). Let $\phi \in \Phi$. Taking $Q = B_2^n$ in formula (3.14) and together with Proposition 3.3, one has

$$\Omega_\phi^{orlicz}(K) \leq G_\phi^{orlicz}(K) = \inf_{Q \in \mathcal{K}_0} \{nV_\phi(K, \text{vrad}(Q^\circ)Q)\} \leq nV_\phi(K, B_2^n) = S_\phi(K).$$

(ii). Let $\phi \in \Psi$. Taking $Q = B_2^n$ in formula (3.15) and together with Proposition 3.3, one has

$$\Omega_\phi^{orlicz}(K) \geq G_\phi^{orlicz}(K) = \sup_{Q \in \mathcal{K}_0} \{nV_\phi(K, \text{vrad}(Q^\circ)Q)\} \geq nV_\phi(K, B_2^n) = S_\phi(K).$$

Proposition 3.5 Let $K \in \mathcal{K}_0$ be a convex body with the origin in its interior.

(i) For $\phi \in \Phi$, one has

$$\Omega_\phi^{orlicz}(K) \leq G_\phi^{orlicz}(K) \leq \phi(\text{vrad}(K^\circ)) \cdot n|K|.$$

(ii) For $\phi \in \Psi$, one has

$$\Omega_\phi^{orlicz}(K) \geq G_\phi^{orlicz}(K) \geq \phi(\text{vrad}(K^\circ)) \cdot n|K|.$$

Proof. (i). Let $\phi \in \Phi$. Taking $Q = K$ in formula (3.14) and by Proposition 3.3, one gets

$$\Omega_\phi^{orlicz}(K) \leq G_\phi^{orlicz}(K) \leq nV_\phi(K, \text{vrad}(K^\circ)K) = \phi(\text{vrad}(K^\circ)) \cdot n|K|,$$

where for $K \in \mathcal{K}_0$ and for $\phi : (0, \infty) \rightarrow (0, \infty)$,

$$nV_\phi(K, \text{vrad}(K^\circ)K) = \int_{S^{n-1}} \phi(\text{vrad}(K^\circ)) h_K(u) dS(K, u) = \phi(\text{vrad}(K^\circ)) \cdot n|K|.$$

(ii). Let $\phi \in \Psi$. Taking $Q = K$ in formula (3.15) and by Proposition 3.3, one gets

$$\Omega_\phi^{orlicz}(K) \geq G_\phi^{orlicz}(K) \geq nV_\phi(K, \text{vrad}(K^\circ)K) = \phi(\text{vrad}(K^\circ)) \cdot n|K|.$$

Remark. Replacing K by its polar body K° , one has, for $\phi \in \Phi$,

$$\Omega_\phi^{orlicz}(K^\circ) \leq G_\phi^{orlicz}(K^\circ) \leq nV_\phi(K^\circ, \text{vrad}(K)K^\circ) = \phi(\text{vrad}(K)) \cdot n|K^\circ|.$$

Hence, for $\phi \in \Phi$,

$$\Omega_\phi^{orlicz}(K)\Omega_\phi^{orlicz}(K^\circ) \leq G_\phi^{orlicz}(K)G_\phi^{orlicz}(K^\circ) \leq \phi(\text{vrad}(K))\phi(\text{vrad}(K^\circ)) \cdot n^2|K| \cdot |K^\circ|.$$

Moreover, if $\phi(t)\phi(s) \leq [\phi(1)]^2$ for all $t, s > 0$ satisfying $st \leq 1$, the following Santaló style inequality holds: for all $K \in \mathcal{K}_s$ or $K \in \mathcal{K}_c$,

$$\begin{aligned} \Omega_\phi^{orlicz}(K)\Omega_\phi^{orlicz}(K^\circ) &\leq G_\phi^{orlicz}(K)G_\phi^{orlicz}(K^\circ) \leq \phi(\text{vrad}(K))\phi(\text{vrad}(K^\circ)) \cdot n^2|K| \cdot |K^\circ| \\ &\leq \phi^2(1)n^2\omega_n^2 = [G_\phi^{orlicz}(B_2^n)]^2 = [\Omega_\phi^{orlicz}(B_2^n)]^2, \end{aligned} \quad (3.17)$$

where the last inequality follows from the Blaschke-Santaló inequality and the first equality follows from Corollary 3.1. For instance, if $\phi(t) = t^p$ with $p \geq 0$, one gets the Santaló style inequality for L_p affine and geominimal surface areas (see e.g., [27, 44, 47, 51]).

Similarly, for $\phi \in \Psi$ with $\phi(t)\phi(s) \geq A[\phi(1)]^2$ for some constant $A > 0$ and for all $t, s > 0$ satisfying $st \leq 1$, one has, for all $K \in \mathcal{K}_s$ or $K \in \mathcal{K}_c$,

$$\begin{aligned}\Omega_\phi^{\text{orlicz}}(K)\Omega_\phi^{\text{orlicz}}(K^\circ) &\geq G_\phi^{\text{orlicz}}(K)G_\phi^{\text{orlicz}}(K^\circ) \geq \phi(\text{vrad}(K))\phi(\text{vrad}(K^\circ)) \cdot n^2|K| \cdot |K^\circ| \\ &\geq Ac^n\phi^2(1)n^2\omega_n^2 = Ac^n[G_\phi^{\text{orlicz}}(B_2^n)]^2 = Ac^n[\Omega_\phi^{\text{orlicz}}(B_2^n)]^2,\end{aligned}$$

where the last inequality follows from the Bourgain-Milman inverse Santaló inequality and the first equality follows from Corollary 3.1. For instance, if $\phi(t) = t^p$ with $-n < p < 0$, one gets the Santaló style inequality for L_p affine and geominimal surface areas [44, 47].

The following theorem deals with the monotonicity for Orlicz L_ϕ affine and geominimal surface areas. We always assume that the function ψ has inverse function $\psi^{-1}(\cdot)$. Let $H(t) = (\phi \circ \psi^{-1})(t)$ be the composition function of ϕ and ψ^{-1} . Recall that all functions $\phi(t) \in \Psi$ are decreasing; and hence if $\phi(t), \psi(t) \in \Psi$, then $H(t)$ is increasing. Similar to the definition for Ψ , we are not interested in the case $H(t)$ being concave decreasing (as otherwise $H(t)$ is eventually a constant function, which leads to ϕ being a constant function). Moreover, let $H(0) = \lim_{t \rightarrow 0} H(t)$ if the limit exists and is finite; on the other hand, let $H(0) = \infty$ if $\lim_{t \rightarrow 0} H(t) = \infty$. Similarly, one can also let $H(\infty) = \lim_{t \rightarrow \infty} H(t)$ if the limit exists and is finite; or simply $H(\infty) = \infty$ if $\lim_{t \rightarrow \infty} H(t) = \infty$.

Theorem 3.1 *Let $K \in \mathcal{K}_0$ be a convex body with the origin in its interior.*

(i) *Assume that ϕ and ψ satisfy one of the following conditions: (a) $\phi \in \Phi$ and $\psi \in \Psi$ with $H(t)$ increasing; (b) $\phi, \psi \in \Phi$ with $H(t)$ decreasing; (c) $H(t)$ concave increasing with either $\phi, \psi \in \Phi$ or $\phi, \psi \in \Psi$. Then*

$$\frac{\Omega_\phi^{\text{orlicz}}(K)}{n|K|} \leq H\left(\frac{\Omega_\psi^{\text{orlicz}}(K)}{n|K|}\right) \quad \text{and} \quad \frac{G_\phi^{\text{orlicz}}(K)}{n|K|} \leq H\left(\frac{G_\psi^{\text{orlicz}}(K)}{n|K|}\right).$$

(ii) *Assume that ϕ and ψ satisfy one of the following conditions: (d) $\phi \in \Psi$ and $\psi \in \Phi$ with $H(t)$ increasing; (e) $H(t)$ convex decreasing with one in Φ and another one in Ψ ; (f) $H(t)$ convex increasing with either $\phi, \psi \in \Phi$ or $\phi, \psi \in \Psi$. Then*

$$\frac{\Omega_\phi^{\text{orlicz}}(K)}{n|K|} \geq H\left(\frac{\Omega_\psi^{\text{orlicz}}(K)}{n|K|}\right) \quad \text{and} \quad \frac{G_\phi^{\text{orlicz}}(K)}{n|K|} \geq H\left(\frac{G_\psi^{\text{orlicz}}(K)}{n|K|}\right).$$

Remark. When $\phi(t) = t^p$ and $\psi(t) = t^q$ for some $p, q \neq -n, 0$, one recovers the monotonicity properties for the L_p affine and geominimal surface areas (see e.g., [27, 44, 47, 50]). Clearly, condition (a) is equivalent to condition (d) if both $\phi(t)$ and $\psi(t)$ have inverse functions. If $H(t)$ is increasing and $\phi^{-1}(t), \psi^{-1}(t)$ both exist, condition (c) is equivalent to condition (f). This follows from the following claim: if $H(t)$ (and hence $H^{-1}(t)$) is increasing, $H(t)$ and $H^{-1}(t)$ have different convexity and concavity. In fact, without loss of generality, assume that $H(t)$ is increasing and convex. For all $t, s \in (0, \infty)$ and $\lambda \in [0, 1]$, we have,

$$\begin{aligned}H^{-1}(\lambda t + (1 - \lambda)s) &= H^{-1}(\lambda H(H^{-1}(t)) + (1 - \lambda)H(H^{-1}(s))) \\ &\geq H^{-1}(H(\lambda H^{-1}(t) + (1 - \lambda)H^{-1}(s))) = \lambda H^{-1}(t) + (1 - \lambda)H^{-1}(s),\end{aligned}$$

which leads to the concavity of $H^{-1}(t)$. Along the same line, one can also prove that the concavity of $H(t)$ implies the convexity of $H^{-1}(t)$. (A similar argument shows that if H is decreasing, then $H(t)$ and $H^{-1}(t)$ have the same convexity and concavity.)

Proof. We only prove the case for Orlicz L_ϕ geominimal surface area. The proof for $\Omega_\phi^{orlicz}(K)$ goes along the same line and hence is omitted.

(i). For condition (a) $\phi \in \Phi$ and $\psi \in \Psi$ with $H(t)$ increasing: Proposition 3.5 and the increasing property of $H(t)$ imply that

$$\frac{G_\phi^{orlicz}(K)}{n|K|} \leq \phi(\text{vrad}(K^\circ)) = H(\psi(\text{vrad}(K^\circ))) \leq H\left(\frac{G_\psi^{orlicz}(K)}{n|K|}\right).$$

For condition (b) $\phi, \psi \in \Phi$ with $H(t)$ decreasing: Proposition 3.5 and the decreasing property of $H(t)$ imply that

$$\frac{G_\phi^{orlicz}(K)}{n|K|} \leq \phi(\text{vrad}(K^\circ)) = H(\psi(\text{vrad}(K^\circ))) \leq H\left(\frac{G_\psi^{orlicz}(K)}{n|K|}\right).$$

For condition (c): The concavity of $H(t)$ with Jensen's inequality imply that, $\forall K, Q \in \mathcal{K}_0$,

$$\begin{aligned} \frac{V_\phi(K, \text{vrad}(Q^\circ)Q)}{|K|} &= \frac{1}{n|K|} \int_{S^{n-1}} H\left[\psi\left(\frac{\text{vrad}(Q^\circ)h_Q(u)}{h_K(u)}\right)\right] h_K(u) dS(K, u) \\ &\leq H\left[\frac{1}{n|K|} \int_{S^{n-1}} \psi\left(\frac{\text{vrad}(Q^\circ)h_Q(u)}{h_K(u)}\right) h_K(u) dS(K, u)\right] \\ &= H\left(\frac{V_\psi(K, \text{vrad}(Q^\circ)Q)}{|K|}\right). \end{aligned}$$

Let $H(t)$ be increasing and concave with $\phi, \psi \in \Phi$. By formula (3.14), one has

$$\begin{aligned} \frac{G_\phi^{orlicz}(K)}{n|K|} &= \inf_{Q \in \mathcal{K}_0} \frac{nV_\phi(K, \text{vrad}(Q^\circ)Q)}{n|K|} \leq \inf_{Q \in \mathcal{K}_0} H\left(\frac{nV_\psi(K, \text{vrad}(Q^\circ)Q)}{n|K|}\right) \\ &= H\left(\inf_{Q \in \mathcal{K}_0} \frac{nV_\psi(K, \text{vrad}(Q^\circ)Q)}{n|K|}\right) = H\left(\frac{G_\psi^{orlicz}(K)}{n|K|}\right). \end{aligned}$$

Let $H(t)$ be increasing and concave with $\phi, \psi \in \Psi$. By formula (3.15), one has

$$\frac{G_\phi^{orlicz}(K)}{n|K|} = \sup_{Q \in \mathcal{K}_0} \frac{nV_\phi(K, \text{vrad}(Q^\circ)Q)}{n|K|} \leq H\left(\sup_{Q \in \mathcal{K}_0} \frac{nV_\psi(K, \text{vrad}(Q^\circ)Q)}{n|K|}\right) = H\left(\frac{G_\psi^{orlicz}(K)}{n|K|}\right).$$

(ii). For condition (d) $\phi \in \Psi$ and $\psi \in \Phi$ with $H(t)$ increasing: Proposition 3.5 and the increasing property of $H(t)$ imply that

$$\frac{G_\phi^{orlicz}(K)}{n|K|} \geq \phi(\text{vrad}(K^\circ)) = H(\psi(\text{vrad}(K^\circ))) \geq H\left(\frac{G_\psi^{orlicz}(K)}{n|K|}\right).$$

For condition (e): the convexity of $H(t)$ together with Jensen's inequality imply that

$$\begin{aligned} \frac{V_\phi(K, \text{vrad}(Q^\circ)Q)}{|K|} &= \frac{1}{n|K|} \int_{S^{n-1}} H\left[\psi\left(\frac{\text{vrad}(Q^\circ)h_Q(u)}{h_K(u)}\right)\right] h_K(u) dS(K, u) \\ &\geq H\left(\frac{V_\psi(K, \text{vrad}(Q^\circ)Q)}{|K|}\right). \end{aligned} \tag{3.18}$$

Let $\phi \in \Psi$ and $\psi \in \Phi$ with $H(t)$ convex decreasing: taking the supremum over $Q \in \mathcal{K}_0$, together with the decreasing property of $H(t)$ and formulas (3.14) and (3.15), one has

$$\frac{G_\phi^{\text{Orlicz}}(K)}{n|K|} \geq \sup_{Q \in \mathcal{K}_0} H\left(\frac{nV_\psi(K, \text{vrad}(Q^\circ)Q)}{n|K|}\right) = H\left(\inf_{Q \in \mathcal{K}_0} \frac{nV_\psi(K, \text{vrad}(Q^\circ)Q)}{n|K|}\right) = H\left(\frac{G_\psi^{\text{Orlicz}}(K)}{n|K|}\right).$$

Similarly, for $\phi \in \Phi$ and $\psi \in \Psi$ with $H(t)$ convex decreasing:

$$\frac{G_\phi^{\text{Orlicz}}(K)}{n|K|} \geq \inf_{Q \in \mathcal{K}_0} H\left(\frac{nV_\psi(K, \text{vrad}(Q^\circ)Q)}{n|K|}\right) = H\left(\sup_{Q \in \mathcal{K}_0} \frac{nV_\psi(K, \text{vrad}(Q^\circ)Q)}{n|K|}\right) = H\left(\frac{G_\psi^{\text{Orlicz}}(K)}{n|K|}\right).$$

For condition (f) $\phi, \psi \in \Phi$ with $H(t)$ convex increasing: taking the infimum over $Q \in \mathcal{K}_0$ in inequality (3.18), together with formula (3.14), one has

$$\frac{G_\phi^{\text{Orlicz}}(K)}{n|K|} \geq \inf_{Q \in \mathcal{K}_0} H\left(\frac{nV_\psi(K, \text{vrad}(Q^\circ)Q)}{n|K|}\right) = H\left(\inf_{Q \in \mathcal{K}_0} \frac{nV_\psi(K, \text{vrad}(Q^\circ)Q)}{n|K|}\right) = H\left(\frac{G_\psi^{\text{Orlicz}}(K)}{n|K|}\right).$$

Similarly, for $\phi, \psi \in \Psi$ with $H(t)$ convex increasing: taking the supremum over $Q \in \mathcal{K}_0$ in inequality (3.18), together with formula (3.15), one has

$$\frac{G_\phi^{\text{Orlicz}}(K)}{n|K|} \geq \sup_{Q \in \mathcal{K}_0} H\left(\frac{nV_\psi(K, \text{vrad}(Q^\circ)Q)}{n|K|}\right) = H\left(\sup_{Q \in \mathcal{K}_0} \frac{nV_\psi(K, \text{vrad}(Q^\circ)Q)}{n|K|}\right) = H\left(\frac{G_\psi^{\text{Orlicz}}(K)}{n|K|}\right).$$

Let $\psi = t \in \Phi$ and $H(t) = \phi(t) \in \Phi$ be concave increasing. Part (i) of Theorem 3.1 implies

$$\frac{\Omega_\phi^{\text{Orlicz}}(K)}{n|K|} \leq \phi\left(\frac{\Omega_1^{\text{Orlicz}}(K)}{n|K|}\right) \quad \text{and} \quad \frac{G_\phi^{\text{Orlicz}}(K)}{n|K|} \leq \phi\left(\frac{G_1^{\text{Orlicz}}(K)}{n|K|}\right). \quad (3.19)$$

If $\psi = t \in \Phi$ and $\phi(t) = H(t) \in \Psi$ is convex decreasing, part (ii) of Theorem 3.1 implies

$$\frac{\Omega_\phi^{\text{Orlicz}}(K)}{n|K|} \geq \phi\left(\frac{\Omega_1^{\text{Orlicz}}(K)}{n|K|}\right) \quad \text{and} \quad \frac{G_\phi^{\text{Orlicz}}(K)}{n|K|} \geq \phi\left(\frac{G_1^{\text{Orlicz}}(K)}{n|K|}\right). \quad (3.20)$$

We now prove the following affine isoperimetric inequality for Orlicz L_ϕ affine and geominimal surface areas. Write B_K for the origin-symmetric Euclidean ball with $|B_K| = |K|$ for $K \in \mathcal{K}_0$.

Theorem 3.2 *Let $K \in \mathcal{K}_c$ or $K \in \mathcal{K}_s$.*

(i) *For $\phi \in \Phi$, the following affine isoperimetric inequality holds, with equality if and only if K is an origin-symmetric ellipsoid,*

$$\Omega_\phi^{\text{Orlicz}}(K) \leq G_\phi^{\text{Orlicz}}(K) \leq G_\phi^{\text{Orlicz}}([B_{K^\circ}]^\circ) = \Omega_\phi^{\text{Orlicz}}([B_{K^\circ}]^\circ).$$

If in addition ϕ is increasing, then

$$\Omega_\phi^{\text{Orlicz}}(K) \leq G_\phi^{\text{Orlicz}}(K) \leq G_\phi^{\text{Orlicz}}(B_K) = \Omega_\phi^{\text{Orlicz}}(B_K),$$

with equality if K is an origin-symmetric ellipsoid. Moreover, if ϕ is strictly increasing, equality holds if and only if K is an origin-symmetric ellipsoid.

(ii) *For $\phi \in \Psi$, the following affine isoperimetric inequality holds with equality if K is an origin-symmetric ellipsoid,*

$$\Omega_\phi^{\text{Orlicz}}(K) \geq G_\phi^{\text{Orlicz}}(K) \geq G_\phi^{\text{Orlicz}}(B_K) = \Omega_\phi^{\text{Orlicz}}(B_K).$$

If in addition ϕ is strictly decreasing, equality holds if and only if K is an origin-symmetric ellipsoid.

Proof. First, Blaschke-Santaló inequality implies that, for all $K \in \mathcal{K}_c$ or $K \in \mathcal{K}_s$,

$$\text{vrad}(K)\text{vrad}(K^\circ) \leq 1 = \text{vrad}(B_K)\text{vrad}([B_K]^\circ) = \text{vrad}(K)\text{vrad}([B_K]^\circ).$$

Therefore, for all $K \in \mathcal{K}_c$ or $K \in \mathcal{K}_s$, one has

$$\text{vrad}(K^\circ) \leq \text{vrad}([B_K]^\circ), \quad (3.21)$$

with equality if and only if K is an origin-symmetric ellipsoid.

(i). Let $\phi \in \Phi$. Corollary 3.1 and Proposition 3.5 imply

$$\begin{aligned} \Omega_\phi^{\text{orlicz}}(K) &\leq G_\phi^{\text{orlicz}}(K) \leq \phi(\text{vrad}(K^\circ)) \cdot n|K| = \phi(\text{vrad}(B_{K^\circ})) \cdot n|[B_{K^\circ}]^\circ| \cdot \frac{|K|}{|[B_{K^\circ}]^\circ|} \\ &= G_\phi^{\text{orlicz}}([B_{K^\circ}]^\circ) \frac{|K||K^\circ|}{|[B_{K^\circ}]^\circ||B_{K^\circ}|} \leq G_\phi^{\text{orlicz}}([B_{K^\circ}]^\circ) = \Omega_\phi^{\text{orlicz}}([B_{K^\circ}]^\circ), \end{aligned}$$

where the last inequality follows from the Blaschke-Santaló inequality. Clearly, equality holds if K is an origin-symmetric ellipsoid. On the other hand, equality holds only if equality holds in the Blaschke-Santaló inequality, that is, K has to be an origin-symmetric ellipsoid.

Assume in addition that $\phi \in \Phi$ is increasing. Inequality (3.21) together with Corollary 3.1 and Proposition 3.5 imply

$$\Omega_\phi^{\text{orlicz}}(K) \leq G_\phi^{\text{orlicz}}(K) \leq n|K|\phi(\text{vrad}(K^\circ)) \leq n|B_K|\phi(\text{vrad}([B_K]^\circ)) = G_\phi^{\text{orlicz}}(B_K) = \Omega_\phi^{\text{orlicz}}(B_K),$$

with equality if K is an origin-symmetric ellipsoid. Moreover, if ϕ is strictly increasing, equality holds only if equality holds in inequality (3.21), i.e., K has to be an origin-symmetric ellipsoid.

(ii). Let $\phi \in \Psi$. Recall that $\phi \in \Psi$ is decreasing. Inequality (3.21) together with Corollary 3.1 and Proposition 3.5 imply

$$\Omega_\phi^{\text{orlicz}}(K) \geq G_\phi^{\text{orlicz}}(K) \geq n|K|\phi(\text{vrad}(K^\circ)) \geq n|B_K|\phi(\text{vrad}([B_K]^\circ)) = G_\phi^{\text{orlicz}}(B_K) = \Omega_\phi^{\text{orlicz}}(B_K),$$

with equality if K is an origin-symmetric ellipsoid. Moreover if ϕ is strictly decreasing, equality holds only if equality holds in inequality (3.21), i.e., K has to be an origin-symmetric ellipsoid.

Remark. Part (i) of Theorem 3.2 asserts that *among all convex bodies in \mathcal{K}_c (or \mathcal{K}_s) with fixed volume, Orlicz L_ϕ affine and geominimal surface areas for $\phi \in \Phi$ attain the maximum at origin-symmetric ellipsoids*. Moreover, if ϕ is also strictly increasing, the origin-symmetric ellipsoids are the only maximizers. Similarly, part (ii) of Theorem 3.2 asserts that *among all convex bodies in \mathcal{K}_c (or \mathcal{K}_s) with fixed volume, Orlicz L_ϕ affine and geominimal surface areas for strictly decreasing function $\phi \in \Psi$ attain the minimum at and only at origin-symmetric ellipsoids*. When $\phi(t) = t^p$ with $-n \neq p \in \mathbb{R}$, one recovers the L_p affine isoperimetric inequalities for the L_p affine and geominimal surface areas (see e.g., [27, 35, 36, 44, 47]).

The following result removes the condition on the centroid (or the Santaló point) of K in Theorem 3.2 for certain ϕ . The case $\phi(t) = t^p$ with $p \in (-n, 1)$ were proved in [47, 50]. See also [48] for similar results on general affine surface areas.

Corollary 3.2 *Let $K \in \mathcal{K}_0$ be a convex body with the origin in its interior.*

(i). *Let $\phi \in \Phi$ be a concave increasing function. The Orlicz L_ϕ affine and geominimal surface areas both attain the maximum at origin-symmetric ellipsoids among all convex bodies in \mathcal{K}_0 with fixed volume. More precisely,*

$$\Omega_\phi^{\text{orlicz}}(K) \leq G_\phi^{\text{orlicz}}(K) \leq G_\phi^{\text{orlicz}}(B_K) = \Omega_\phi^{\text{orlicz}}(B_K)$$

with equality if K is an origin-symmetric ellipsoid. Moreover, if ϕ is increasing and strictly concave, equality holds if and only if K is an origin-symmetric ellipsoid.

(ii). Let $\phi \in \Psi$ be a convex decreasing function. The Orlicz L_ϕ affine and geominimal surface areas both attain the minimum at origin-symmetric ellipsoids among all convex bodies in \mathcal{K}_0 with fixed volume. More precisely,

$$\Omega_\phi^{\text{Orlicz}}(K) \geq G_\phi^{\text{Orlicz}}(K) \geq G_\phi^{\text{Orlicz}}(B_K) = \Omega_\phi^{\text{Orlicz}}(B_K)$$

with equality if K is an origin-symmetric ellipsoid. Moreover, if ϕ is decreasing and strictly convex, equality holds if and only if K is an origin-symmetric ellipsoid.

Proof. We only prove the case for Orlicz L_ϕ geominimal surface area. The proof for Orlicz L_ϕ affine surface area goes along the same line.

(i). Let $\phi \in \Phi$ be a concave increasing function. Assume that z_0 is the centroid of K . By inequality (3.19) and the translation invariance of $G_1^{\text{Orlicz}}(K)$, one has,

$$\frac{G_\phi^{\text{Orlicz}}(K)}{n|K|} \leq \phi\left(\frac{G_1^{\text{Orlicz}}(K)}{n|K|}\right) = \phi\left(\frac{G_1^{\text{Orlicz}}(K - z_0)}{n|K - z_0|}\right) \leq \phi\left(\frac{G_1^{\text{Orlicz}}(B_K)}{n|B_K|}\right) = \phi(\text{vrad}([B_K]^\circ)), \quad (3.22)$$

where the second inequality follows from Theorem 3.2 (as $K - z_0 \in \mathcal{K}_c$) and the last equality follows from Corollary 3.1 (with function $\phi(t) = t$). Therefore, again by Corollary 3.1, one has

$$G_\phi^{\text{Orlicz}}(K) \leq \phi(\text{vrad}([B_K]^\circ)) \cdot n|K| = \phi(\text{vrad}([B_K]^\circ)) \cdot n|B_K| = G_\phi^{\text{Orlicz}}(B_K).$$

Clearly, equality holds if K is an origin-symmetric ellipsoid.

Assume in addition that ϕ is strictly concave. To have equality in inequality (3.22), one requires $K - z_0$ to be an origin-symmetric ellipsoid. We now claim that $z_0 = 0$. To this end, assume that, without loss of generality, $K = z_0 + rB_2^n$ is a Euclidean ball with center z_0 and radius $r > 0$. Moreover, $\|z_0\| < r$ and

$$\int_{S^{n-1}} h_{z_0 + rB_2^n}(u) d\sigma(u) = nr|B_2^n|.$$

Hence, for strictly concave function $\phi \in \Phi$,

$$\begin{aligned} G_\phi^{\text{Orlicz}}(K) &= G_\phi^{\text{Orlicz}}(z_0 + rB_2^n) \leq nV_\phi(z_0 + rB_2^n, B_2^n) \\ &= n\omega_n r^n \int_{S^{n-1}} \phi\left(\frac{1}{h_{z_0 + rB_2^n}(u)}\right) \frac{h_{z_0 + rB_2^n}(u)}{nr\omega_n} d\sigma(u) \\ &\leq n\omega_n r^n \cdot \phi\left(\int_{S^{n-1}} \frac{h_{z_0 + rB_2^n}(u)}{nr\omega_n h_{z_0 + rB_2^n}(u)} d\sigma(u)\right) \\ &= n\omega_n r^n \cdot \phi(1/r) = G_\phi^{\text{Orlicz}}(rB_2^n), \end{aligned} \quad (3.23)$$

where the last inequality follows from Jensen's inequality (for concave function ϕ) and the last equality follows from Corollary 3.1. To have equality in inequality (3.23), one requires, in particular, equality for Jensen's inequality. That is, $h_{z_0 + rB_2^n}(u)$ is a constant on S^{n-1} due to the strict concavity of ϕ , and hence $z_0 = 0$ as desired.

(ii). Let $\phi \in \Psi$ be convex decreasing. Assume that z_0 is the centroid of K . By inequality (3.20) and the translation invariance of $G_1^{\text{Orlicz}}(K)$, one has,

$$\phi^{-1}\left(\frac{G_\phi^{\text{Orlicz}}(K)}{n|K|}\right) \leq \frac{G_1^{\text{Orlicz}}(K)}{n|K|} = \frac{G_1^{\text{Orlicz}}(K - z_0)}{n|K - z_0|} \leq \frac{G_1^{\text{Orlicz}}(B_K)}{n|B_K|} = \text{vrad}([B_K]^\circ),$$

where the second inequality follows from Theorem 3.2 (as $K - z_0 \in \mathcal{K}_c$) and the last equality follows from Corollary 3.1 (with function $\phi(t) = t$). As ϕ is decreasing and by Corollary 3.1, one has

$$\frac{G_\phi^{orlicz}(K)}{n|K|} \geq \phi(\text{vrad}([B_K]^\circ)) \iff G_\phi^{orlicz}(K) \geq \phi(\text{vrad}([B_K]^\circ))n|K| = G_\phi^{orlicz}(B_K). \quad (3.24)$$

Clearly, equality holds if K is an origin-symmetric ellipsoid.

Assume in addition that ϕ is convex and strictly decreasing. To have equality in inequality (3.24), one requires $K - z_0$ to be an origin-symmetric ellipsoid, which implies $z_0 = 0$. In fact, similar to the case (i), one has

$$\begin{aligned} G_\phi^{orlicz}(K) &= G_\phi^{orlicz}(z_0 + rB_2^n) = \sup_{Q \in \mathcal{K}_0} \{nV_\phi(z_0 + rB_2^n, \text{vrad}(Q^\circ)Q)\} \\ &\geq n\omega_n r^{n-1} \int_{S^{n-1}} \phi\left(\frac{1}{h_{z_0+rB_2^n}(u)}\right) \frac{h_{z_0+rB_2^n}(u)}{n\omega_n} d\sigma(u) \\ &\geq n\omega_n r^n \cdot \phi(1/r) = G_\phi^{orlicz}(rB_2^n). \end{aligned} \quad (3.25)$$

To have equality in (3.25), one requires, in particular, equality for Jensen's inequality. That is, $h_{z_0+rB_2^n}(u)$ is a constant on S^{n-1} due to the strict convexity of ϕ , and hence $z_0 = 0$ as desired.

4 Orlicz mixed L_ϕ affine and geominimal surface areas for multiple convex bodies

In this section, the Orlicz mixed L_ϕ affine and geominimal surface areas for multiple convex bodies and their basic properties are briefly discussed. We will omit most of the proofs because these proofs are either similar to those for single convex body discussed in Section 3 or similar to those for mixed p -affine and mixed L_p geominimal surface areas in [27, 45, 49].

We use $\vec{\phi}$ for $(\phi_1, \phi_2, \dots, \phi_n)$. We say $\vec{\phi} \in \Phi^n$ (or $\vec{\phi} \in \Psi^n$) if each $\phi_i \in \Phi$ (or $\phi_i \in \Psi$). Similarly, $\mathbf{L} = (L_1, \dots, L_n) \in \mathcal{S}_0^n$ and $\mathbf{K} = (K_1, \dots, K_n) \in \mathcal{F}_0^n$ mean that each $L_i \in \mathcal{S}_0$ and each $K_i \in \mathcal{F}_0$ respectively. We use \mathbf{L}° for $(L_1^\circ, \dots, L_n^\circ)$. Define $V_{\vec{\phi}}(\mathbf{K}, \mathbf{Q})$ for $\mathbf{K} \in \mathcal{F}_0^n$ and $\mathbf{Q} \in \mathcal{K}_0^n$ by

$$V_{\vec{\phi}}(\mathbf{K}, \mathbf{Q}) = \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^n \left[\phi_i \left(\frac{h_{Q_i}(u)}{h_{K_i}(u)} \right) h_{K_i}(u) f_{K_i}(u) \right]^{\frac{1}{n}} d\sigma(u).$$

When $\mathbf{K} \in \mathcal{F}_0^n$ and $\mathbf{L} \in \mathcal{S}_0^n$, we use the following formula

$$V_{\vec{\phi}}(\mathbf{K}, \mathbf{L}^\circ) = \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^n \left[\phi_i \left(\frac{1}{\rho_{L_i}(u) h_{K_i}(u)} \right) h_{K_i}(u) f_{K_i}(u) \right]^{\frac{1}{n}} d\sigma(u).$$

When $\phi_i = \phi$, $K_i = K$ and $L_i = L$ for all $i = 1, 2, \dots, n$, one gets $V_{\vec{\phi}}(\mathbf{K}; \mathbf{L}^\circ) = V_\phi(K, L^\circ)$.

We now propose our definition for the Orlicz mixed L_ϕ affine surface area.

Definition 4.1 Let $K_1, \dots, K_n \in \mathcal{F}_0$.

(i) For $\vec{\phi} \in \Phi^n$, we define $\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K})$ by

$$\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K}) = \inf_{\mathbf{L} \in \mathcal{S}_0^n} \{nV_{\vec{\phi}}(\mathbf{K}; \mathbf{L}^\circ) \text{ with } |L_1| = |L_2| = \dots = |L_n| = \omega_n\}.$$

(ii) For $\vec{\phi} \in \Psi^n$, we define $\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K})$ by

$$\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K}) = \sup_{\mathbf{L} \in \mathcal{S}_0^n} \{nV_{\vec{\phi}}(\mathbf{K}; \mathbf{L}^\circ) \text{ with } |L_1| = |L_2| = \cdots = |L_n| = \omega_n\}.$$

Similarly, the Orlicz mixed L_ϕ geominimal surface area can be defined as follows.

Definition 4.2 Let $K_1, \dots, K_n \in \mathcal{F}_0$.

(i) For $\vec{\phi} \in \Phi^n$, we define $G_{\vec{\phi}}^{orlicz}(\mathbf{K})$ by

$$G_{\vec{\phi}}^{orlicz}(\mathbf{K}) = \inf_{\mathbf{Q} \in \mathcal{K}_0^n} \{nV_{\vec{\phi}}(\mathbf{K}; \mathbf{Q}) \text{ with } |Q_1^\circ| = |Q_2^\circ| = \cdots = |Q_n^\circ| = \omega_n\}.$$

(ii) For $\vec{\phi} \in \Psi^n$, we define $G_{\vec{\phi}}^{orlicz}(\mathbf{K})$ by

$$G_{\vec{\phi}}^{orlicz}(\mathbf{K}) = \sup_{\mathbf{Q} \in \mathcal{K}_0^n} \{nV_{\vec{\phi}}(\mathbf{K}; \mathbf{Q}) \text{ with } |Q_1^\circ| = |Q_2^\circ| = \cdots = |Q_n^\circ| = \omega_n\}.$$

Remark. The case $\vec{\phi} = (t^p, t^p, \dots, t^p)$ corresponds to the mixed p -affine and mixed L_p geominimal surface areas (see e.g., [27, 45, 49]). For the geominimal case, several different mixed L_p geominimal surface areas could be proposed for the same \mathbf{K} . Analogously, one can also define several mixed L_ϕ Orlicz affine and geominimal surface areas; however, due to high similarity of their properties (as one can see in [49]), we only focus on the one by Definition 4.2 in this paper.

The following result could be proved by a similar argument to Proposition 3.3.

Proposition 4.1 Let $K_1, \dots, K_n \in \mathcal{F}_0$.

(i) For $\vec{\phi} \in \Phi^n$, one has

$$\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K}) \leq G_{\vec{\phi}}^{orlicz}(\mathbf{K}).$$

(ii) For $\vec{\phi} \in \Psi^n$, one has

$$\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K}) \geq G_{\vec{\phi}}^{orlicz}(\mathbf{K}).$$

The following result states that Orlicz mixed L_ϕ affine and geominimal surface areas are affine invariant. The proof is similar to that of Proposition 3.2. For $T \in SL(n)$ and $\mathbf{K} = (K_1, \dots, K_n) \in \mathcal{K}_0^n$, we let $T\mathbf{K} = (TK_1, \dots, TK_n)$.

Proposition 4.2 Let $\mathbf{K} \in \mathcal{F}_0^n$. For $\vec{\phi} \in \Phi^n$ or $\vec{\phi} \in \Psi^n$, one has

$$\Omega_{\vec{\phi}}^{orlicz}(T\mathbf{K}) = \Omega_{\vec{\phi}}^{orlicz}(\mathbf{K}); \quad G_{\vec{\phi}}^{orlicz}(T\mathbf{K}) = G_{\vec{\phi}}^{orlicz}(\mathbf{K}), \quad \forall T \in SL(n).$$

The classical Alexander-Fenchel inequality for the mixed volume (see [37]) is one of the most important inequalities in convex geometry. It has been extended to the mixed p -affine surface area and the mixed L_p geominimal surface area (see e.g., [25, 27, 45, 49, 52]). See also [46] for similar inequalities related to general mixed affine surface areas. Here, we prove the following Alexander-Fenchel type inequality for Orlicz mixed L_ϕ affine and geominimal surface areas.

Theorem 4.1 Let $\mathbf{K} \in \mathcal{F}_0^n$. For $\vec{\phi} \in \Phi^n$ or $\vec{\phi} \in \Psi^n$, one has

$$[\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K})]^n \leq \prod_{i=1}^n \Omega_{\phi_i}^{orlicz}(K_i) \quad \text{and} \quad [G_{\vec{\phi}}^{orlicz}(\mathbf{K})]^n \leq \prod_{i=1}^n G_{\phi_i}^{orlicz}(K_i).$$

Moreover, if $\vec{\phi} \in \Psi^n$, the following Alexander-Fenchel type inequality holds: for $1 \leq m \leq n$,

$$\begin{aligned} [\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K})]^m &\leq \prod_{i=0}^{m-1} \Omega_{(\phi_1, \dots, \phi_{n-m}, \phi_{n-i}, \dots, \phi_{n-i})}^{orlicz}(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_m), \\ [G_{\vec{\phi}}^{orlicz}(\mathbf{K})]^m &\leq \prod_{i=0}^{m-1} G_{(\phi_1, \dots, \phi_{n-m}, \phi_{n-i}, \dots, \phi_{n-i})}^{orlicz}(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_m). \end{aligned}$$

Remark. If each $\phi_i \in \Phi$ satisfies $\phi_i(t)\phi_i(s) \leq [\phi(1)]^2$ for all $t, s > 0$ with $st \leq 1$, then

$$\begin{aligned} [\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K})\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K}^\circ)]^n &\leq [G_{\vec{\phi}}^{orlicz}(\mathbf{K})G_{\vec{\phi}}^{orlicz}(\mathbf{K}^\circ)]^n \leq \prod_{i=1}^n [G_{\phi_i}^{orlicz}(K_i)G_{\phi_i}^{orlicz}(K_i^\circ)] \\ &\leq \prod_{i=1}^n [G_{\phi_i}^{orlicz}(B_2^n)]^2 = \prod_{i=1}^n [\Omega_{\phi_i}^{orlicz}(B_2^n)]^2, \end{aligned}$$

where the last inequality follows from inequality (3.17).

Proof. We only prove the geominimal case and omit the proof for affine case. Let

$$\begin{aligned} H_0(u) &= \prod_{i=1}^{n-m} \left[\phi_i \left(\frac{h_{Q_i}(u)}{h_{K_i}(u)} \right) h_{K_i}(u) f_{K_i}(u) \right]^{\frac{1}{n}}, \\ H_{i+1}(u) &= \left[\phi_{n-i} \left(\frac{h_{Q_{n-i}}(u)}{h_{K_{n-i}}(u)} \right) h_{K_{n-i}}(u) f_{K_{n-i}}(u) \right]^{\frac{1}{n}}, \quad i = 0, \dots, m-1. \end{aligned}$$

Hölder's inequality (see [17]) implies

$$\begin{aligned} [V_{\vec{\phi}}(\mathbf{K}; \mathbf{Q})]^m &= \left(\frac{1}{n} \int_{S^{n-1}} H_0(u) H_1(u) \cdots H_m(u) d\sigma(u) \right)^m \\ &\leq \prod_{i=0}^{m-1} \left(\frac{1}{n} \int_{S^{n-1}} H_0(u) [H_{i+1}(u)]^m d\sigma(u) \right) = \prod_{i=0}^{m-1} A_i, \end{aligned} \quad (4.26)$$

where for $i = 0, 1, \dots, m-1$,

$$A_i = V_{(\phi_1, \dots, \phi_{n-m}, \phi_{n-i}, \dots, \phi_{n-i})}(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_m; Q_1, \dots, Q_{n-m}, \underbrace{Q_{n-i}, \dots, Q_{n-i}}_m).$$

In particular, if $m = n$, inequality (4.26) implies that

$$[V_{\vec{\phi}}(\mathbf{K}; \mathbf{Q})]^n \leq \prod_{i=1}^n V_{\phi_i}(K_i, Q_i).$$

Let $\vec{\phi} \in \Phi^n$. By Definitions 3.2 and 4.2, one has

$$\begin{aligned} [G_{\vec{\phi}}^{orlicz}(\mathbf{K})]^n &= \left[\inf_{\mathbf{Q} \in \mathcal{K}_0^n} \{nV_{\vec{\phi}}(\mathbf{K}; \mathbf{Q}) \text{ with } |Q_1^\circ| = |Q_2^\circ| = \cdots = |Q_n^\circ| = \omega_n\} \right]^n \\ &\leq \inf_{\mathbf{Q} \in \mathcal{K}_0^n} \left\{ \prod_{i=1}^n [nV_{\phi_i}(K_i, Q_i)] \text{ with } |Q_1^\circ| = \cdots = |Q_n^\circ| = \omega_n \right\} \\ &= \prod_{i=1}^n \inf_{Q_i \in \mathcal{K}_0^n} \{nV_{\phi_i}(K_i, Q_i) \text{ with } |Q_i^\circ| = \omega_n\} = \prod_{i=1}^n G_{\phi_i}^{orlicz}(K_i). \end{aligned}$$

Similarly, if $\phi \in \Psi^n$, one gets

$$[G_{\vec{\phi}}^{orlicz}(\mathbf{K})]^n \leq \prod_{i=1}^n \sup_{Q_i \in \mathcal{K}_0^n} \{nV_{\phi_i}(K_i, Q_i) \text{ with } |Q_i^\circ| = \omega_n\} = \prod_{i=1}^n G_{\phi_i}^{orlicz}(K_i).$$

Moreover, inequality (4.26) implies that for $\vec{\phi} \in \Psi^n$ and $1 \leq m \leq n$,

$$[G_{\vec{\phi}}^{orlicz}(\mathbf{K})]^m \leq \prod_{i=0}^{m-1} \sup_{\mathbf{Q} \in \mathcal{K}_0^n} \{A_i \text{ with } |Q_1^\circ| = \dots = |Q_n^\circ| = \omega_n\},$$

and the desired Alexander-Fenchel type inequality follows if one notices

$$\sup_{\mathbf{Q} \in \mathcal{K}_0^n} \{A_i : |Q_1^\circ| = \dots = |Q_n^\circ| = \omega_n\} \leq G_{(\phi_1, \dots, \phi_{n-m}, \phi_{n-i}, \dots, \phi_{n-i})}^{orlicz}(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_m).$$

The following result is a direct consequence from Theorem 4.1 and Propositions 3.4 and 4.1.

Corollary 4.1 *Let $K_1, \dots, K_n \in \mathcal{F}_0$. For $\vec{\phi} \in \Phi^n$, one has*

$$[\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K})]^n \leq [G_{\vec{\phi}}^{orlicz}(\mathbf{K})]^n \leq S_{\phi_1}(K_1) \cdots S_{\phi_n}(K_n).$$

A direct consequence of Proposition 4.1, and Theorems 3.2 and 4.1 is the following isoperimetric type inequality, which holds for any possible combinations of \mathcal{F}_c and \mathcal{F}_s . Due to Corollary 3.2, K_i can be assumed even in \mathcal{F}_0 if $\phi_i \in \Phi$ is concave increasing.

Theorem 4.2 *Let $K_i \in \mathcal{F}_c$ or $K_i \in \mathcal{F}_s$. For $\vec{\phi} \in \Phi^n$, one has*

$$[\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K})]^n \leq [G_{\vec{\phi}}^{orlicz}(\mathbf{K})]^n \leq \prod_{i=1}^n G_{\phi_i}^{orlicz}([B_{(K_i)^\circ}]^\circ) = \prod_{i=1}^n \Omega_{\phi_i}^{orlicz}([B_{(K_i)^\circ}]^\circ).$$

If in addition all ϕ_i are increasing, then

$$[\Omega_{\vec{\phi}}^{orlicz}(\mathbf{K})]^n \leq [G_{\vec{\phi}}^{orlicz}(\mathbf{K})]^n \leq \prod_{i=1}^n G_{\phi_i}^{orlicz}(B_{K_i}) = \prod_{i=1}^n \Omega_{\phi_i}^{orlicz}(B_{K_i}).$$

In literature, the i -th mixed p -affine surface area and i -th mixed L_p geominimal surface area are of interest, see details in e.g., [25, 41, 45, 49, 52]. We will define and briefly discuss properties and inequalities for the Orlicz i -th mixed L_ϕ affine and geominimal surface areas. Let $K, L \in \mathcal{F}_0^+$ and $Q_1, Q_2 \in \mathcal{K}_0$, we define $V_{\phi_1, \phi_2, i}(K, L; Q_1, Q_2)$ for $i \in \mathbb{R}$ by

$$nV_{\phi_1, \phi_2, i}(K, L; Q_1, Q_2) = \int_{S^{n-1}} \left[\phi_1 \left(\frac{h_{Q_1}(u)}{h_K(u)} \right) h_K(u) f_K(u) \right]^{\frac{n-i}{n}} \left[\phi_2 \left(\frac{h_{Q_2}(u)}{h_L(u)} \right) h_L(u) f_L(u) \right]^{\frac{i}{n}} d\sigma(u).$$

For $L_1, L_2 \in \mathcal{S}_0$, we use the following formula

$$nV_{\phi_1, \phi_2, i}(K, L; L_1^\circ, L_2^\circ) = \int_{S^{n-1}} \left[\phi_1 \left(\frac{\rho_{L_1}(u)^{-1}}{h_K(u)} \right) h_K(u) f_K(u) \right]^{\frac{n-i}{n}} \left[\phi_2 \left(\frac{\rho_{L_2}(u)^{-1}}{h_L(u)} \right) h_L(u) f_L(u) \right]^{\frac{i}{n}} d\sigma(u).$$

Hölder's inequality (see [17]) implies

$$\begin{aligned} [V_{\phi_1, \phi_2, i}(K, L; Q_1, Q_2)]^n &\leq [V_{\phi_1}(K, Q_1)]^{n-i} [V_{\phi_2}(L, Q_2)]^i, \quad \text{if } 0 < i < n; \\ [V_{\phi_1, \phi_2, i}(K, L; Q_1, Q_2)]^n &\geq [V_{\phi_1}(K, Q_1)]^{n-i} [V_{\phi_2}(L, Q_2)]^i, \quad \text{if } i < 0 \text{ or } i > n. \end{aligned} \quad (4.27)$$

We now propose our definition for the Orlicz i -th mixed L_ϕ affine surface area for $K, L \in \mathcal{F}_0^+$.

Definition 4.3 Let $K, L \in \mathcal{F}_0^+$ and $i \in \mathbb{R}$.

(i) For $\phi_1, \phi_2 \in \Phi$,

$$\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L) = \inf_{\{L_1, L_2 \in \mathcal{S}_0\}} \{nV_{\phi_1, \phi_2, i}(K, L; L_1^\circ, L_2^\circ) : |L_1| = |L_2| = \omega_n\}.$$

(ii) For $\phi_1, \phi_2 \in \Psi$,

$$\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L) = \sup_{\{L_1, L_2 \in \mathcal{S}_0\}} \{nV_{\phi_1, \phi_2, i}(K, L; L_1^\circ, L_2^\circ) : |L_1| = |L_2| = \omega_n\}.$$

Similarly, the Orlicz i -th mixed L_ϕ geominimal surface area can be defined as follows.

Definition 4.4 Let $K, L \in \mathcal{F}_0^+$ and $i \in \mathbb{R}$.

(i) For $\phi_1, \phi_2 \in \Phi$,

$$G_{\phi_1, \phi_2, i}^{orlicz}(K, L) = \inf_{\{Q_1, Q_2 \in \mathcal{K}_0\}} \{nV_{\phi_1, \phi_2, i}(K, L; Q_1^\circ, Q_2^\circ) : |Q_1| = |Q_2| = \omega_n\}.$$

(ii) For $\phi_1, \phi_2 \in \Psi$,

$$G_{\phi_1, \phi_2, i}^{orlicz}(K, L) = \sup_{\{Q_1, Q_2 \in \mathcal{K}_0\}} \{nV_{\phi_1, \phi_2, i}(K, L; Q_1^\circ, Q_2^\circ) : |Q_1| = |Q_2| = \omega_n\}.$$

Clearly, $\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L) = \Omega_{\phi_1, \phi_2, n-i}^{orlicz}(L, K)$ and $G_{\phi_1, \phi_2, i}^{orlicz}(K, L) = G_{\phi_1, \phi_2, n-i}^{orlicz}(L, K)$ for all $i \in \mathbb{R}$ and all $K, L \in \mathcal{F}_0^+$. Moreover,

$$\Omega_{\phi_1, \phi_2, 0}^{orlicz}(K, L) = \Omega_{\phi_1}^{orlicz}(K) \quad \& \quad \Omega_{\phi_1, \phi_2, n}^{orlicz}(K, L) = \Omega_{\phi_2}^{orlicz}(L); \quad (4.28)$$

$$G_{\phi_1, \phi_2, 0}^{orlicz}(K, L) = G_{\phi_1}^{orlicz}(K) \quad \& \quad G_{\phi_1, \phi_2, n}^{orlicz}(K, L) = G_{\phi_2}^{orlicz}(L). \quad (4.29)$$

One can see that the Orlicz i -th mixed L_ϕ affine and geominimal surface areas are all affine invariant. Moreover, for $K, L \in \mathcal{F}_0^+$ and $i \in \mathbb{R}$ one has

$$\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L) \leq G_{\phi_1, \phi_2, i}^{orlicz}(K, L), \quad \phi_1, \phi_2 \in \Phi; \quad (4.30)$$

$$\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L) \geq G_{\phi_1, \phi_2, i}^{orlicz}(K, L), \quad \phi_1, \phi_2 \in \Psi. \quad (4.31)$$

Theorem 4.3 Let $K, L \in \mathcal{F}_0^+$ and $i < j < k$. For $\phi_1, \phi_2 \in \Psi$, one has

$$\begin{aligned} [\Omega_{\phi_1, \phi_2, j}^{orlicz}(K, L)]^{k-i} &\leq [\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^{k-j} [\Omega_{\phi_1, \phi_2, k}^{orlicz}(K, L)]^{j-i}; \\ [G_{\phi_1, \phi_2, j}^{orlicz}(K, L)]^{k-i} &\leq [G_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^{k-j} [G_{\phi_1, \phi_2, k}^{orlicz}(K, L)]^{j-i}. \end{aligned}$$

Proof. We only prove the case for geominimal and omit the proof for affine case. Let $K, L \in \mathcal{F}_0^+$ and $Q_1, Q_2 \in \mathcal{K}_0$. Let $i < j < k$ which implies $0 < \frac{k-j}{k-i} < 1$. Note also $k-i > 0, k-j > 0$ and $j-i > 0$. Hölder's inequality implies that,

$$V_{\phi_1, \phi_2, j}(K, L; Q_1, Q_2) \leq [V_{\phi_1, \phi_2, i}(K, L; Q_1, Q_2)]^{\frac{k-j}{k-i}} [V_{\phi_1, \phi_2, k}(K, L; Q_1, Q_2)]^{\frac{j-i}{k-i}}.$$

The desired result follows by taking the supremum over $Q_1, Q_2 \in \mathcal{K}_0$ with $|Q_1^\circ| = |Q_2^\circ| = \omega_n$.

Remark. Let $\phi_1, \phi_2 \in \Psi$. For $0 < i < n$, let $(i, j, k) = (0, i, n)$ in Theorem 4.3, by formulas (4.28) and (4.29), we have

$$[\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n \leq [\Omega_{\phi_1}^{orlicz}(K)]^{n-i} [\Omega_{\phi_2}^{orlicz}(L)]^i \quad \& \quad [G_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n \leq [G_{\phi_1}^{orlicz}(K)]^{n-i} [G_{\phi_2}^{orlicz}(L)]^i.$$

Equality always hold if $i = 0$ or $i = n$. Similarly, for $i < 0$ or $i > n$, one has

$$\begin{aligned} [\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n &\geq [\Omega_{\phi_1}^{orlicz}(K)]^{n-i} [\Omega_{\phi_2}^{orlicz}(L)]^i, \\ [G_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n &\geq [G_{\phi_1}^{orlicz}(K)]^{n-i} [G_{\phi_2}^{orlicz}(L)]^i. \end{aligned} \quad (4.32)$$

Theorem 4.4 Let $K, L \in \mathcal{F}_0^+$ be convex bodies with centroid (or Santaló point) at the origin.
(i) For $0 \leq i \leq n$ and $\phi_1, \phi_2 \in \Phi$, one has

$$[\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n \leq [G_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n \leq [G_{\phi_1}^{orlicz}([B_{K^\circ}]^\circ)]^{n-i} [G_{\phi_2}^{orlicz}([B_{L^\circ}]^\circ)]^i.$$

If in addition ϕ_1, ϕ_2 are both increasing, then

$$[\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n \leq [G_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n \leq [G_{\phi_1}^{orlicz}(B_K)]^{n-i} [G_{\phi_2}^{orlicz}(B_L)]^i.$$

(ii) Let \mathcal{E} be an origin-symmetric ellipsoid. For $\phi_1, \phi_2 \in \Psi$ and $i \leq 0$,

$$[\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, \mathcal{E})]^n \geq [G_{\phi_1, \phi_2, i}^{orlicz}(K, \mathcal{E})]^n \geq [G_{\phi_1}^{orlicz}(B_K)]^{n-i} [G_{\phi_2}^{orlicz}(\mathcal{E})]^i.$$

Proof. (i). Let $\phi_1, \phi_2 \in \Phi$ and $0 \leq i \leq n$. Taking the infimum from both sides of inequality (4.27) over $Q_1, Q_2 \in \mathcal{K}_0$ with $|Q_1^\circ| = |Q_2^\circ| = \omega_n$ and by Definitions 3.2 and 4.4, one has

$$[G_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n \leq [G_{\phi_1}^{orlicz}(K)]^{n-i} [G_{\phi_2}^{orlicz}(L)]^i.$$

Combining with Theorem 3.2 and inequality (4.30), one gets, for $0 \leq i \leq n$ and $\phi_1, \phi_2 \in \Phi$,

$$\begin{aligned} [\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n &\leq [G_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n \leq [G_{\phi_1}^{orlicz}(K)]^{n-i} [G_{\phi_2}^{orlicz}(L)]^i \\ &\leq [G_{\phi_1}^{orlicz}([B_{K^\circ}]^\circ)]^{n-i} [G_{\phi_2}^{orlicz}([B_{L^\circ}]^\circ)]^i. \end{aligned}$$

If in addition ϕ_1, ϕ_2 are both increasing, then

$$\begin{aligned} [\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n &\leq [G_{\phi_1, \phi_2, i}^{orlicz}(K, L)]^n \leq [G_{\phi_1}^{orlicz}(K)]^{n-i} [G_{\phi_2}^{orlicz}(L)]^i \\ &\leq [G_{\phi_1}^{orlicz}(B_K)]^{n-i} [G_{\phi_2}^{orlicz}(B_L)]^i. \end{aligned}$$

(ii). Let $i \leq 0$ and $\phi_1, \phi_2 \in \Psi$. Note that both ϕ_1 and ϕ_2 are decreasing. Inequalities (4.31) and (4.32) together with Theorem 3.2 imply

$$\begin{aligned} [\Omega_{\phi_1, \phi_2, i}^{orlicz}(K, \mathcal{E})]^n &\geq [G_{\phi_1, \phi_2, i}^{orlicz}(K, \mathcal{E})]^n \geq [G_{\phi_1}^{orlicz}(K)]^{n-i} [G_{\phi_2}^{orlicz}(\mathcal{E})]^i \\ &\geq [G_{\phi_1}^{orlicz}(B_K)]^{n-i} [G_{\phi_2}^{orlicz}(\mathcal{E})]^i. \end{aligned}$$

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Deping Ye, deping.ye@mun.ca
 Department of Mathematics and Statistics
 Memorial University of Newfoundland
 St. John's, Newfoundland, Canada A1C 5S7